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# Approximating zeros of monotone operators by proximal point algorithms

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**Abstract** In this paper, we introduce two kinds of iterative algorithms for the problem of finding zeros of maximal monotone operators. Weak and strong convergence theorems are established in a real Hilbert space. As applications, we consider a problem of finding a minimizer of a convex function.

**Keywords** Nonexpansive mapping · Fixed point · Monotone operator · Zero · Proximal point algorithm

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## 1 Introduction

Throughout this paper, we assume that $H$ is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $T$ be a set-valued mapping.

(a) The set $D(T)$ defined by

$$D(T) = \{u \in H : T(u) \neq \emptyset\}$$

is called the effective domain of $T$;

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(b) The set $R(T)$ defined by

$$R(T) = \bigcup_{u \in H} T(u)$$

is called the range of $T$;

(c) The set $G(T)$ defined by

$$G(T) = \{(u, v) \in H \times H : u \in D(T), v \in R(T)\}$$

is said to be the graph of $T$.

Recall that a mapping $T$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (u, x), (v, y) \in G(T).$$

$T$ is said to be maximal monotone if it is not properly contained in any other monotone operator. The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone mappings, see [1–26] and the references therein.

In this paper, we consider the problem of finding zeros of maximal monotone operators by the proximal point algorithm. To be more precise, we introduce two kinds of iterative schemes. Weak and strong convergence theorems are established in a real Hilbert space. As applications, we also consider a problem of finding a minimizer of a convex function in the Sect. 4.

## 2 Preliminaries

Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$. In this paper, we always assume that $T : C \to 2^H$ is a maximal monotone operator. A classical method to solve the following set-valued equation

$$0 \in Tx, \tag{2.1}$$

is the proximal point algorithm. To be more precise, start with any point $x_0 \in H$, and update $x_{n+1}$ iteratively conforming to the following recursion

$$x_n \in x_{n+1} + \lambda_n T x_{n+1}, \quad n \geq 0, \tag{2.2}$$

where $\{\lambda_n\} \subset [\lambda, \infty)$, $(\lambda > 0)$, is a sequence of real numbers. However, as pointed in [10], the ideal form of the method is often impractical since, in many cases, to solve the problem (2.2) exactly is either impossible or the same difficult as the original problem (2.1). Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of $T$.

In 1976, Rockafellar [21] gave an inexact variant of the method

$$x_0 \in H, \quad x_n + e_{n+1} \in x_{n+1} + \lambda_n T x_{n+1}, \quad n \geq 0, \tag{2.3}$$

where $\{e_n\}$ is regarded as an error sequence. This an inexact proximal point algorithm. It was shown that, if

$$\sum_{n=0}^{\infty} \|e_n\| < \infty,$$

then the sequence $\{x_n\}$ defined by ( 2.3) converges weakly to a zero of $T$ provided that $T^{-1}(0) \neq \emptyset$. In [11], Güler obtained an example to show that Rockafellar's proximal point algorithm (2.3) does not converge strongly, in general.

Recently, many authors studied the problems of modifying Rockafellar's proximal point algorithm so that strong convergence is guaranteed. Cho et al. [6] proved the following result.

**Theorem CKZ-1** *Let $H$ be a real Hilbert space, $\Omega$ a nonempty closed convex subset of $H$, and $T : \Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $P_\Omega$ be the metric projection of $H$ onto $\Omega$. Suppose that, for any given $x_n \in H$, $\lambda_n > 0$ and $e_n \in H$, there exists $\bar{x}_n \in \Omega$ conforming to the following set-valued mapping equation*

$$x_n + e_n \in \bar{x}_n + \lambda_n T\bar{x}_n,$$

*where $\{\lambda_n\} \subset (0, +\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \|e_n\|^2 < \infty.$$

*Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such that*

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*For any fixed $u \in \Omega$, define the sequence $\{x_n\}$ iteratively as follows:*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_\Omega(\bar{x}_n - e_n), \quad n \geq 0.$$

*Then $\{x_n\}$ converges strongly to a fixed point $z$ of $T$, where $z = \lim_{t\to\infty} J_t u$.*

They also obtained the following weak convergence theorem.

**Theorem CKZ-2** *Let $H$ be a real Hilbert space, $\Omega$ a nonempty closed convex subset of $H$, and $T : \Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $P_\Omega$ be the metric projection of $H$ onto $\Omega$. Suppose that, for any given $x_n \in H$, $\lambda_n > 0$ and $e_n \in H$, there exists $\bar{x}_n \in \Omega$ conforming to the following set-valued mapping equation*

$$x_n + e_n \in \bar{x}_n + \lambda_n T\bar{x}_n,$$

*where* $\liminf \lambda_n > 0$ *and*

$$\sum_{n=0}^{\infty} \|e_n\|^2 < \infty.$$

*Let $\{\alpha_n\}$ be a real sequences in $[0, 1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and define a sequence $\{x_n\}$ iteratively as follows:*

$$x_0 \in \Omega, \quad x_{n+1} = \alpha_n x_n + \beta_n P_C(\bar{x}_n - e_n), \quad n \geq 0.$$

*Then the sequence $\{x_n\}$ converges weakly to a zero point $x^*$ of $T$.*

In this paper, motivated by the research work going on in this direction, we continue to consider the problem of finding a zero of the maximal monotone operator $T$. Weak and strong convergence theorems are established under mild restrictions imposed on the error sequence $\{e_n\}$ comparing with the restriction studied by Cho et al. [6]. The results presented in this paper improve the corresponding results announced by many others.

In order to prove our main result, we need the following lemmas.

The first Lemma can be derived from Eckstein [10, Lemma 2] immediately.

**Lemma 2.1** *Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$. For any given $x_n \in H$, $\lambda_n > 0$, and $e_n \in H$, there exists $\bar{x}_n \in C$ conforming to the following set-valued mapping equation (in short, SVME):*

$$x_n + e_n \in \bar{x}_n + \lambda_n T \bar{x}_n. \tag{2.4}$$

*Furthermore, for any $p \in T^{-1}(0)$, we have*

$$\langle x_n - \bar{x}_n, x_n - \bar{x}_n + e_n \rangle \leq \langle x_n - p, x_n - \bar{x}_n + e_n \rangle$$

*and*

$$\|\bar{x}_n - e_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2.$$

**Lemma 2.2** (Liu [16]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of positive numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

*where $\{t_n\}$ is a sequence in $[0, 1]$. Assume that the following conditions are satisfied*

(i) $t_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=0}^{\infty} t_n = \infty$;

(ii) $b_n = \circ(t_n)$;

(iii) $\sum_{n=0}^{\infty} c_n < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.3** (Tan and Xu [24]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

*If $\sum_{n=0}^{\infty} b_n < \infty$, then the limit of $\{a_n\}$ exists.*

**Lemma 2.4** (Browder [1]) *Let $E$ be a uniformly convex Banach space, $C$ be a nonempty closed convex subset of $E$ and $S : C \to C$ be a non-expansive mapping. Then $I - S$ is demi-closed at zero.*

**Lemma 2.5** (Cho et al. [27]) *Let $E$ be a uniformly convex Banach space and $B_r(0)$ be a closed ball of $E$. Then there exists a continuous strictly increasing convex function $g\colon [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

*for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.*

## 3 Main results

**Theorem 3.1** *Let $H$ be a real Hilbert space, $C$ a nonempty, closed and convex subset of $H$ and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $P_C$ be a metric projection from $H$ onto $C$. For any $x_n \in H$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$ conforming to the SVME (2.4), where $\{\lambda_n\} \subset (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$ with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + \beta_n P_C(\bar{x}_n - e_n) + \gamma P_C f_n, \quad n \geq 0, \tag{$\Upsilon$}$$

*where* $u \in C$ *is a fixed point and* $\{f_n\}$ *is a bounded sequence in* $H$. *Then the sequence* $\{x_n\}$ *generated by* $(\Upsilon)$ *converges strongly to a zero point* $z$ *of* $T$, *where* $z = \lim_{t\to\infty} J_t u$, *if and only if* $e_n \to 0$ *as* $n \to \infty$.

*Proof* First, show that the necessity. Assume that $x_n \to z$ as $n \to \infty$, where $z \in T^{-1}(0)$. It follows from (2.4) that

$$\begin{aligned}\|\bar{x}_n - z\| &\leq \|x_n - z\| + \|e_n\|\\ &\leq \|x_n - z\| + \eta_n\|x_n - \bar{x}_n\|\\ &\leq (1+\eta_n)\|x_n - z\| + \eta_n\|\bar{x}_n - z\|.\end{aligned}$$

This implies that

$$\|\bar{x}_n - z\| \leq \frac{1+\eta_n}{1-\eta_n}\|x_n - z\|.$$

It follows that $\bar{x}_n \to z$ as $n \to \infty$. Note that

$$\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\| \leq \eta_n(\|x_n - z\| + \|z - \bar{x}_n\|).$$

This shows that $e_n \to 0$ as $n \to \infty$.

Next, we show the sufficiency. The proof is divided into three steps.

**Step 1** Show that $\{x_n\}$ is bounded.

From the assumptions $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$ and $\sup_{n\geq 0} \eta_n = \eta < 1$, we see

$$\|e_n\| \leq \|x_n - \bar{x}_n\|.$$

For any $p \in T^{-1}(0)$. It follows from Lemma 2.1 that

$$\begin{aligned}\|P_C(\bar{x}_n - e_n) - p\|^2 &\leq \|\bar{x}_n - e_n - p\|^2\\ &\leq \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2\\ &\leq \|x_n - p\|^2.\end{aligned}$$

That is,

$$\|P_C(\bar{x}_n - e_n) - p\| \leq \|x_n - p\|. \tag{3.1}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n u + \beta_n P_C(\bar{x}_n - e_n) + \gamma_n P_C f_n - p\|\\ &\leq \alpha_n\|u - p\| + \beta_n\|P_C(\bar{x}_n - e_n) - p\| + \gamma_n\|P_C f_n - p\|\\ &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|f_n - p\|.\end{aligned} \tag{3.2}$$

Putting

$$M = \max\{\|x_0 - p\|, \|u - p\|, \sup_{n\geq 0} \|f_n - p\|\},$$

we show that $\|x_n - p\| \leq M$ for all $n \geq 0$. It is easy to see that the result holds for $n = 0$. Assume that the result holds for some $n \geq 0$. Next, we prove that $\|x_{n+1} - p\| \leq M$. Indeed, from (3.2), we see that

$$\|x_{n+1} - p\| \leq M.$$

This shows that the sequence $\{x_n\}$ is bounded.

**Step 2** Show that $\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \leq 0$, where $z = \lim_{t\to\infty} J_t u$. The existence of limit $J_t u$ is guaranteed by Lemma 1 of Bruck [2].

Since $T$ is maximal monotone, $T_t u \in T J_t u$ and $T_{\lambda_n} x_n \in T J_{\lambda_n} x_n$, we see

$$\begin{aligned}\langle u - J_t u, J_{\lambda_n} x_n - J_t u\rangle &= -t\langle T_t u, J_t u - J_{\lambda_n} x_n\rangle \\ &= -t\langle T_t u - T_{\lambda_n} x_n, J_t u - J_{\lambda_n} x_n\rangle - t\langle T_{\lambda_n} x_n, J_t u - J_{\lambda_n} x_n\rangle \\ &\leq -\frac{t}{\lambda_n}\langle x_n - J_{\lambda_n} x_n, J_t u - J_{\lambda_n} x_n\rangle.\end{aligned}$$

Since $\lambda_n \to \infty$ as $n \to \infty$, for any $t > 0$, we have

$$\limsup_{n\to\infty}\langle u - J_t u, J_{\lambda_n} x_n - J_t u\rangle \leq 0. \tag{3.3}$$

On the other hand, by the nonexpansivity of $J_{\lambda_n}$, we obtain

$$\|J_{\lambda_n}(x_n + e_n) - J_{\lambda_n} x_n\| \leq \|(x_n + e_n) - x_n\| = \|e_n\|.$$

From the assumption $e_n \to 0$ as $n \to \infty$ and (3.3), we arrive at

$$\limsup_{n\to\infty}\langle u - J_t u, J_{\lambda_n}(x_n + e_n) - J_t u\rangle \leq 0. \tag{3.4}$$

From (2.4), we see that

$$\|P_C(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\| \leq \|(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\| \leq \|e_n\|.$$

That is,

$$\lim_{n\to\infty} \|P_C(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\| = 0. \tag{3.5}$$

Combining (3.4) with (3.5), we arrive at

$$\limsup_{n\to\infty}\langle u - J_t u, P_C(\bar{x}_n - e_n) - J_t u\rangle \leq 0. \tag{3.6}$$

On the other hand, from the algorithm ($\Upsilon$), we see that

$$x_{n+1} - P_C(\bar{x}_n - e_n) = \alpha_n[u - P_C(\bar{x}_n - e_n)] + \gamma_n[P_C f_n - P_C(\bar{x}_n - e_n)].$$

It follows from the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$ that

$$x_{n+1} - P_C(\bar{x}_n - e_n) \to 0 \quad \text{as } n \to \infty,$$

which combines with (3.6) yields that

$$\limsup_{n\to\infty}\langle u - J_t u, x_{n+1} - J_t u\rangle \leq 0, \quad \forall t \geq 0. \tag{3.7}$$

From $z = \lim_{t\to\infty} J_t u$ and (3.7), we can obtain that

$$\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \le 0. \tag{3.8}$$

**Step 3** Show that $x_n \to z$ as $n \to \infty$.
Note that

$$\begin{aligned} x_{n+1} - z &= \alpha_n u + \beta_n P_C(\bar{x}_n - e_n) + \gamma_n P_C f_n - z \\ &= (1-\alpha_n)[P_C(\bar{x}_n - e_n) - z] + \alpha_n(u - z) \\ &\quad + \gamma_n[P_C f_n - P_C(\bar{x}_n - e_n)] \end{aligned} \tag{3.9}$$

It follows from (3.1) and (3.9) that

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad = \langle (1-\alpha_n)[P_C(\bar{x}_n - e_n) - z] + \alpha_n(u-z) + \gamma_n[P_C f_n - P_C(\bar{x}_n - e_n)], x_{n+1} - z\rangle \\ &\quad = (1-\alpha_n)\langle P_C(\bar{x}_n - e_n) - z, x_{n+1} - z\rangle + \alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\qquad + \gamma_n\langle P_C f_n - P_C(\bar{x}_n - e_n), x_{n+1} - z\rangle \\ &\quad \le (1-\alpha_n)\|P_C(\bar{x}_n - e_n) - z\|\|x_{n+1} - z\| + \alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\qquad + \gamma_n\|P_C f_n - P_C(\bar{x}_n - e_n)\|\|x_{n+1} - z\| \\ &\quad \le (1-\alpha_n)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\qquad + \gamma_n\|f_n - (\bar{x}_n - e_n)\|\|x_{n+1} - z\| \\ &\quad \le \frac{1-\alpha_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\qquad + \gamma_n\|f_n - (\bar{x}_n - e_n)\|\|x_{n+1} - z\| \\ &\quad \le \frac{1-\alpha_n}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle u - z, x_{n+1} - z\rangle + \frac{B}{2}\gamma_n, \end{aligned}$$

where $B$ is an appropriate constant such that $2B \ge \sup_{n\ge 0}\{\|f_n - (\bar{x}_n - e_n)\|\|x_{n+1} - z\|\}$. This implies that

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle + \gamma_n B. \tag{3.10}$$

Putting $\sigma_n = \max\{\langle u - z, x_{n+1} - z\rangle, 0\}$, we see that $\sigma \to 0$ as $n \to \infty$. Put $a_n = \|a_n - z\|^2$, $b_n = 2\alpha_n\sigma_n$ and $c_n = \gamma_n B$ for each $n \ge 0$. It follows from (3.10) that

$$a_{n+1} \le (1-\alpha_n)a_n + b_n + c_n.$$

In view of Lemma 2.2, we obtain that $a_n \to 0$ as $n \to \infty$. This shows that $x_n \to z$ as $n \to \infty$. This completes the proof. □

*Remark 3.2* The maximal monotonicity of $T$ is only used to guarantee the existence of solutions of SVME (2.4) for any give $x_n \in H$ and $\lambda_n > 0$. If we assume that $T : C \to 2^H$ is monotone (need not be maximal) and satisfies the range condition:

$$\overline{D(T)} = C \subset \cap_{r>0} R(I + rT).$$

We can see that Theorem 3.1 still holds.

**Corollary 3.3** *Let $H$ be a real Hilbert space, $C$ a nonempty, closed and convex subset of $H$ and $S : C \to C$ a demi-continuous pseudo-contraction with a fixed point $C$. Let $P_C$ be a* *metric projection from $H$ onto $C$. For any $x_n \in C$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$ such that*

$$x_n + e_n = (1 + \lambda_n)\bar{x}_n - \lambda_n S\bar{x}_n, \tag{3.11}$$

*where $\{\lambda_n\} \subset (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$ with $\sup_{n \geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n P_C(\bar{x}_n - e_n) + \gamma P_C f_n, \quad n \geq 0,$$

*where $u \in C$ is a fixed point and $\{f_n\}$ is a bounded sequence in $H$. If the sequence $\{e_n\}$ satisfies the condition $e_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ converges strongly to a fixed point $z$ of $S$, where $z = \lim_{t\to\infty}[I + t(I - S)]^{-1}u$.*

*Proof* Let $T = I - S$. Then $T : C \to H$ is demi-continuous, monotone and satisfies the range condition:

$$\overline{D(T)} = C \subset \cap_{r>0} R(I + rT).$$

For any $y \in C$, define an operator $G : C \to C$ by

$$Gx = \frac{t}{1+t} Sx + \frac{1}{1+t} y.$$

Then $G$ is demi-continuous and strongly pseudo-continuous. By Lan and Wu [15, Theorem 2.2], we see that $G$ has a unique fixed point $x \in C$. That is,

$$y = x + t(I - S)x.$$

This implies that $y \in R(I + tT)$ for all $t > 0$. In particular, for any give $x_n \in C$ and $\lambda_n > 0$, there exist $\bar{x}_n \in C$ and $e_n \in H$ such that

$$x_n + e_n = \bar{x}_n + \lambda_n T\bar{x}_n, \quad n \geq 0.$$

That is,

$$x_n + e_n = (1 + \lambda_n)\bar{x}_n - \lambda_n S\bar{x}_n.$$

Next, from the proof of Theorem 3.1, we can obtain the desired conclusion immediately. □

From Theorem 3.1, we also have the following result immediately.

**Corollary 3.4** *Let $H$ be a real Hilbert space, $C$ a nonempty, closed and convex subset of $H$ and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $P_C$ be a metric projection from $H$ onto $C$. For any $x_n \in H$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$ conforming to the SVME (2.4), where $\{\lambda_n\} \subset (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$*

*with* $\sup_{n\geq 0}\eta_n = \eta < 1$. *Let* $\{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying the following control conditions:*

$$\lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=0}^{\infty}\alpha_n = \infty.$$

*Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + \beta_n P_C(\bar{x}_n - e_n), \quad n \geq 0,$$

*where* $u \in C$ *is a fixed point. Then the sequence* $\{x_n\}$ *converges strongly to a zero point* $z$ *of* $T$, *where* $z = \lim_{t\to\infty} J_t u$, *if and only if* $e_n \to 0$ *as* $n \to \infty$.

Next, we give a Mann-type iterative algorithm and study the weak convergence of the algorithm.

**Theorem 3.5** *Let* $H$ *be a real Hilbert space,* $C$ *a nonempty, closed and convex subset of* $H$ *and* $T : C \to 2^H$ *a maximal monotone operator with* $T^{-1}(0) \neq \emptyset$. *Let* $P_C$ *be a metric projection from* $H$ *onto* $C$. *For any* $x_n \in C$ *and* $\lambda_n > 0$, *find* $\bar{x}_n \in C$ *and* $e_n \in H$ *conforming to the SVME (2.4), where* $\liminf \lambda_n > 0$ *and* $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$ *with* $\sup_{n\geq 0}\eta_n = \eta < 1$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty}\beta_n > 0$;

(c) $\sum_{n=0}^{\infty}\gamma_n < \infty$.

*Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n P_C(\bar{x}_n - e_n) + \gamma P_C f_n, \quad n \geq 0, \tag{$\Upsilon\Upsilon$}$$

*where* $\{f_n\}$ *is a bounded sequence in* $H$. *Then the sequence* $\{x_n\}$ *generated by* $(\Upsilon\Upsilon)$ *converges weakly to a zero point* $x^*$ *of* $T$.

*Proof* For any $p \in T^{-1}(0)$. It follows from Lemma 2.1 that

$$\begin{aligned}\|P_C(\bar{x}_n - e_n) - p\|^2 &\leq \|\bar{x}_n - e_n - p\|^2\\ &\leq \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2.\end{aligned}$$

From the assumptions $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$, we see that

$$\begin{aligned}\|P_C(\bar{x}_n - e_n) - p\|^2 &\leq \|\bar{x}_n - e_n - p\|^2\\ &\leq \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \eta_n^2\|x_n - \bar{x}_n\|^2\\ &\leq \|x_n - p\|^2 - (1-\eta^2)\|x_n - \bar{x}_n\|^2.\end{aligned}$$

It follows from Lemma 2.5 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + \beta_n P_C(\bar{x}_n - e_n) + \gamma_n P_C f_n - p\|^2\\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|P_C(\bar{x}_n - e_n) - p\|^2 + \gamma_n\|P_C f_n - p\|^2\\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n[\|x_n - p\|^2 - (1-\eta^2)\|x_n - \bar{x}_n\|^2] + \gamma_n\|f_n - p\|^2\\ &\leq \|x_n - p\|^2 - \beta_n(1-\eta^2)\|x_n - \bar{x}_n\|^2 + \gamma_n\|f_n - p\|^2\\ &\leq \|x_n - p\|^2 + \gamma_n\|f_n - p\|^2.\end{aligned} \tag{3.12}$$

From Lemma 2.3, we see that the limit of $\{\|x_n - p\|\}$ exists. We, therefore, obtain that the sequence $\{x_n\}$ is bounded. It follows from (3.12) that

$$\beta_n(1-\eta^2)\|x_n - \bar{x}_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|f_n - p\|^2.$$

From the conditions $\liminf_{n\to\infty} \beta_n > 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$, we arrive at

$$\lim_{n\to\infty} \|x_n - \bar{x}_n\| = 0. \tag{3.13}$$

Note that

$$\begin{aligned} \|x_n - J_{\lambda_n} x_n\| &= \|x_n - \bar{x}_n + \bar{x}_n - J_{\lambda_n} x_n\| \\ &\leq \|x_n - \bar{x}_n\| + \|\bar{x}_n - J_{\lambda_n} x_n\| \\ &\leq (1+\eta_n)\|x_n - \bar{x}_n\|. \end{aligned}$$

In view of (3.13), we obtain that

$$\lim_{n\to\infty} \|x_n - J_{\lambda_n} x_n\| = 0. \tag{3.14}$$

Note that

$$\begin{aligned} \|J_{\lambda_n} x_n - J_1 J_{\lambda_n} x_n\| &= \|T_1 J_{\lambda_n} x_n\| \\ &\leq \inf\{\|w\| : w \in T J_{\lambda_n} x_n\} \\ &\leq \|T_{\lambda_n} x_n\| \\ &= \frac{\|x_n - J_{\lambda_n} x_n\|}{\lambda_n}. \end{aligned}$$

In view of the assumption $\liminf \lambda_n > 0$ and (3.14), we see that

$$\lim_{n\to\infty} \|J_{\lambda_n} x_n - J_1 J_{\lambda_n} x_n\| = 0.$$

Let $x^* \in C$ be a weakly subsequential limit of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x^*$ as $i \to \infty$. From (3.14), we see that $J_{\lambda_{n_i}} x_{n_i}$ also converges weakly to $x^*$. Since $J_1$ is nonexpansive, we can obtain that $x^* \in F(J_1) = T^{-1}(0)$ by Lemma 2.4. The Opial's condition (see [18]) guarantees that the sequence $\{x_n\}$ converges weakly to $x^*$. This completes the proof. □

From the proof of Corollary 3.3 and Theorem 3.5, the following result is not hard to derive.

**Corollary 3.6** *Let $H$ be a real Hilbert space, $C$ a nonempty, closed and convex subset of $H$ and $S : C \to C$ a demi-continuous pseudo-contraction with a fixed point in $C$. Let $P_C$ be a metric projection from $H$ onto $C$. For any $x_n \in C$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$*

$$x_n + e_n = (1+\lambda_n)\bar{x}_n - \lambda_n S\bar{x}_n, \quad n \geq 0,$$

*where $\liminf \lambda_n > 0$ and $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$ with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty} \beta_n > 0$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n \alpha_n + \beta_n P_C(\bar{x}_n - e_n) + \gamma P_C f_n, \quad n \geq 0,$$

*where $\{f_n\}$ is a bounded sequence in $H$. Then the sequence $\{x_n\}$ converges weakly to a fixed point $x^*$ of $S$.*

From Theorem 3.5, we also have the following result immediately.

**Corollary 3.7** *Let $H$ be a real Hilbert space, $C$ a nonempty, closed and convex subset of $H$ and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $P_C$ be a metric projection from $H$ onto $C$. For any $x_n \in C$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$ conforming to the SVME (2.4), where $\liminf \lambda_n > 0$ and $\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$ with $\sup_{n \geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$ be a real sequences in $[0, 1]$ satisfying $\limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n P_C(\bar{x}_n - e_n), \quad n \geq 0.$$

*Then the sequence $\{x_n\}$ converges weakly to a zero point $x^*$ of $T$.*

## 4 Applications

In this section, as applications of main Theorems 3.1 and 3.5, we consider the problem of finding a minimizer of a convex function $f$.

Let $H$ be a Hilbert space and $f : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential $\partial f$ of $f$ is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y \rangle, \quad z \in H\}, \quad \forall x \in H.$$

**Theorem 4.1** *Let $H$ be a real Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial f(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0, +\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{e_n\}$ a sequence in $H$ such that $\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$ with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\bar{x}_n$ be the solution of the SVME (2.4) with $T$ replacing by $\partial f$. That is, for any given $x_n \in H$,*

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

*Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ \bar{x}_n = argmin_{x\in H}\{f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2\}, \\ x_{n+1} = \alpha_n u + \beta_n(\bar{x}_n - e_n) + \gamma f_n, \quad n \geq 0, \end{cases}$$

*where $u \in H$ is a fixed point and $\{f_n\}$ is a bounded sequence in $H$. If the sequence $\{e_n\}$ satisfies the condition $e_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ converges strongly to a minimizer of $f$ nearest to $u$.*

*Proof* Since $f : H \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, we have that the subdifferential $\partial f$ of $f$ is maximal monotone by Rockafellar [21]. Notice that

$$\bar{x}_n = \text{argmin}_{x\in H}\left\{f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2\right\}$$

is equivalent to the following

$$0 \in \partial f(\bar{x}_n) + \frac{1}{\lambda_n}(\bar{x}_n - x_n - e_n).$$

It follows that

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

By using Theorem 3.1, we can obtain the desired conclusion immediately. □

**Theorem 4.2** *Let $H$ be a real Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial f(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0, +\infty)$ with $\liminf_{n\to\infty} \lambda_n > 0$ and $\{e_n\}$ a sequence in $H$ such that $\|e_n\| \leq \eta_n\|x_n - \bar{x}_n\|$ with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\bar{x}_n$ be the solution of the SVME (2.4) with $T$ replacing by $\partial f$. That is, for any given $x_n \in H$,*

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

*Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty} \beta_n > 0$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ \bar{x}_n = argmin_{x\in H}\left\{f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2\right\}, \\ x_{n+1} = \alpha_n x_n + \beta_n(\bar{x}_n - e_n) + \gamma f_n, \quad n \geq 0, \end{cases}$$

*where $\{f_n\}$ is a bounded sequence in $H$. Then the sequence $\{x_n\}$ converges weakly to a minimizer of $f$.*

*Proof* We can obtain the desired conclusion easily from the proof of Theorems 3.5 and 4.1. □

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